Practice Problem Set 1, Date: 4 December, 2024

Instructor: Ankan Kar

Divisibility in Number Theory

1. Find the largest integer n such that $n^2 + 4n + 5$ is divisible by $n + 1$.

2. Prove that $7^n - 4^n$ is divisible by 3 for all positive integers n.

3. If p is a prime, prove that $p^2 + p + 1$ is never divisible by 3.

4. Find all integers x such that $3x + 1 \mid 2x^2 + x + 7$.

5. Prove that $2^{2n} - 1$ is divisible by 3 for all positive integers n.

Geometry on Triangles

6. Prove that the medians of a triangle intersect at a single point and divide each median in a ratio of $2 : 1$.

7. In a triangle ABC, if $\angle A = 90^\circ$, show that the length of the median from A is equal to half the length of the hypotenuse.

8. Prove that the sum of the lengths of any two sides of a triangle is greater than the third side.

9. If the incenter I of a triangle is equidistant from all three sides, prove that the triangle is equilateral.

10. In $\triangle ABC$, the lengths of the sides are a, b, c, and the circumradius is R. Show that $a^2 + b^2 + c^2 = 4R^2 + 4r^2 + s^2$, where r is the inradius and s is the semi-perimeter.

Basic Inequalities

11. Prove that for all positive real numbers a, b, c: $a^2 + b^2 + c^2 \geq ab + bc + ca$.

12. If $x, y > 0$, prove that $\frac{x}{y} + \frac{y}{x} \geq 2$.

13. Prove that $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$ for all non-negative a, b.

14. For positive real numbers a, b, c, show that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.

15. Using the AM-GM inequality, prove that $\frac{x^2}{y} + y \geq 2x$ for all positive x, y.

Basic Coordinate Geometry

16. Find the equation of the line passing through the points $(1, 2)$ and $(3, 4)$.

17. Find the distance between the points $(2, -1)$ and $(4, 3)$.

18. Find the midpoint of the segment joining the points $(-1, 5)$ and $(3, 7)$.

19. The line $3x - 4y = 12$ intersects the x-axis and y-axis. Find the coordinates of the intersection points.

20. Prove that the points $(0, 0)$, $(2, 4)$, and $(4, 0)$ form a right triangle.

End